SCHUR-WEYL DUALITY AND THE FROBENIUS FORMULA

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1. Preliminaries

Let S_n denote the symmetric group on n elements, and let $\mathrm{GL}(V)$ denote the set of invertible linear maps $V \to V$, where V is an $n-$dimensional vector space over the field K. Consider the space $V^{\otimes r}$, where r is some integer. The actions of S_n and $\mathrm{GL}(V)$ on $V^{\otimes r}$ commute. When $g \in \mathrm{GL}(V)$ is applied to some tensor $v_1 \otimes \cdots \otimes v_r \in V^{\otimes r}$, it acts tensor-wise, such that $g(v_1 \otimes \cdots \otimes v_r) = g(v_1) \otimes \cdots \otimes g(v_r)$. When $\sigma \in S_n$ acts on the same element, say on the right (as is convention), it permutes the order of the individual tensor positions, such that $(v_1 \otimes \cdots \otimes v_r)\sigma = v_{1\sigma^{-1}} \otimes \cdots \otimes v_{r\sigma^{-1}}$. Clearly these commute.

Let $\Psi : K[\mathrm{GL}(V)] \to \mathrm{End}_K(V^{\otimes r})$ and $\Phi : KS_n \to \mathrm{End}_K(V^{\otimes r})$ be the maps from the group algebras of S_n and $\mathrm{GL}(V)$ induced by the actions given above (these are then representations of S_n and $\mathrm{GL}(V)$). Because these commute, they induce inclusions

$$\Psi(K[GL(V)]) \subseteq \mathrm{End}_{S_n}(V^{\otimes r}) \, , \, \Phi(K[S_n]) \subseteq \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes r}).$$

Schur-Weyl duality is the statement that these inclusions are instead equalities (as per [1]). This result holds for arbitrary infinite fields K, but the classical result proves the case where $K = \mathbb{C}$. In the rest of this paper, we will provide a proof of the classical Schur-Weyl duality via the double centralizer theorem and proving the duality for the Lie algebra $\mathfrak{gl}(V)$. Then, we will discuss irreducible representations of the permutation group in more detail.

2. Double Centralizer Theorem

To prove the titular theorem, we will need the following lemma.

Lemma 2.1. *Let A be any finite dimensional algebra. Then it has only finitely many V_i irreducible representations up to isomorphism, these representations are finitely dimensional, and*

$$A/\operatorname{rad}(A) \cong \bigoplus_i \mathrm{End}(V_i)$$

Proof. This proof is adapted from [2]. To show that an irreducible representation V is finite dimensional, note that $Av \subseteq V$ is a finite dimensional subrepresentation of V for nonzero $v \in V$. As V is irreducible, then $Av = V$, so V is finite dimensional.

Suppose we have r distinct irreducible representations $V_1, \ldots, V_r$. By the Jacobson density theorem, the homomorphism

$$\bigoplus_i \rho_i : A \to \bigoplus_i \mathrm{End}(V_i)$$

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is surjective, implying that $r \leq \sum_i \dim \mathrm{End}(V_i) \leq \dim A$. Thus, A cannot have more than $\dim A$ nonisomorphic irreducible representations. Now take all distinct irreducible representations $V_1, \dots, V_k$. We have that

$$\bigoplus_i \rho_i : A \to \bigoplus_i \mathrm{End}(V_i)$$

is still surjective, such that its kernel is exactly $\mathrm{Rad}(A)$. This proves the statement. ■

Corollary 2.2. *Let A be a finite dimensional algebra with finitely many irreducible representations V_i. A is semisimple if and only if as an algebra,*

$$A \cong \bigoplus_i \mathrm{End}(V_i).$$

Proof. A is semisimple if and only if its radical is zero. ■

We now prove the Double Centralizer Theorem.

Theorem 2.3 (Double Centralizer Theorem)**.** *Let V be a finite-dimensional vector space, let A be a semisimple subalgebra of* $\mathrm{End}(V)$, *and let* $B = \mathrm{End}_A(V)$. *Then B is semisimple,* $A = \mathrm{End}_B(V)$, *and*

$$V \cong \bigoplus_i U_i \otimes W_i,$$

where U_i is a simple submodule of A, *W_i is either* 0 *or a simple submodule of* B, *and i ranges over all possible U_i.*

Proof. This proof is adapted from [3]. As A is semisimple, there exists a decomposition as $A-$modules

$$V \cong \bigoplus U_i \otimes \mathrm{Hom}_A(U_i, V).$$

The action of A on $U_i \otimes \mathrm{Hom}_A(U_i, V)$ is given by $a \cdot (u \otimes v) = (a \cdot u) \otimes v$, which respects the action of A on V. As algebras, A decomposes into

$$A = \bigoplus_i \mathrm{End}(U_i).$$

by Lemma 2.2. Via some algebraic manipulations, we find that

$$\begin{aligned} B &= \mathrm{End}_A(V) \\ &= \mathrm{Hom}_A\left(\bigoplus U_i \otimes W_i, V\right) \\ &= \bigoplus_i \mathrm{Hom}_A(U_i \otimes W_i, V) \\ &= \bigoplus_i \mathrm{Hom}(W_i, \mathrm{Hom}_A(U_i, V)) \\ &= \bigoplus_i \mathrm{End}(W_i). \end{aligned}$$

We now show that W_i is a simple B-module, by showing that any nonzero submodule $W \subset W_i$ is equivalent to W_i.. To do this, we show that for any two elements $f, f' \in W$ there exists a $b \in B$ such that $b \cdot f = f'$, which would imply that W is simple and so $W = W_i$.

Any function $f \in \mathrm{Hom}_A(U_i, V)$ is determined uniquely by the value of $f(u)$, where $u \in U_i$ is nonzero, because U_i is a simple $A-$module. Let $f(u) = v$ and $f'(u) = v'$. By Maschke's

Theorem, V decomposes into $(Av) \otimes W$. Define $T \in \mathrm{End}(V)$ as $T(av) = av'$ on Av and $T(w) = w$ on W. This is an A-hom because it respects the action of A on V, so it is contained in B, and $Tf = f'$, so we are done.

Therefore, by Lemma 2.2 B is semisimple. Doing this construction in reverse by taking W_i to be the simple submodules shows that

$$V \cong \bigoplus_i W_i \otimes \mathrm{Hom}(W_i, V) \cong \bigoplus_i W_i \otimes U_i,$$

which proves the decomposition, and as $U_i \cong \mathrm{Hom}_B(W_i, V)$ we have that $A = \mathrm{End}_B(V)$. This proves the theorem. ■

3. Schur-Weyl Duality

3.1. **Elementary Discussion of Lie Algebras.** A *Lie group* is a group on which multiplication and taking inverses are continuous. A *Lie group homomorphism* is a group homomorphism where the homomorphism is also smooth. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a *Lie bracket* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $[X, X] = 0$ for $X \in \mathfrak{g}$ and

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all elements $X, Y, Z \in \mathfrak{g}$. Any Lie group has an associated Lie algebra.

We are concerned with the Lie algebra $\mathfrak{gl}(V)$, which is simply $\mathrm{End}(V)$ for some finite-dimensional complex vector space V equipped with a commutator as its Lie bracket: $[X, Y] = XY - YX$. It is the asosciated Lie algebra to $\mathrm{GL}(V)$. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We abuse notation similarly to the group representation case, and call V the representation of $\mathfrak{g}$. If V and W are representations of $\mathfrak{g}$, then $V \otimes W$ is also a representation of $\mathfrak{g}$ under $X(u \otimes v) = Xu \otimes v + u \otimes Xv$, for $X \in \mathfrak{g}$.

The universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ encodes data about the representations of $\mathfrak{g}$ in an analogous way to a group algebra RG encoding information about the representations of G. In particular, the hom-set from $\mathcal{U}\mathfrak{g}$ to A as algebras, where A is a $\mathbb{C}$-algebra, is isomorphic to the homset of $\mathfrak{g}$ to $\mathcal{L}\mathfrak{g}$ as Lie algebras, where $\mathcal{L}\mathfrak{g}$ is A as a set with the commutator as the Lie bracket. It is constructed via a quotient of the tensor algebra $\mathcal{T}\mathfrak{g}$. The tensor algebra $\mathcal{T}\mathfrak{g}$ is the set $\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$, where $g^{\otimes 0} = \mathbb{C}$, such that $\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}^{\otimes m} = \mathfrak{g}^{\otimes (n+m)}$. The universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is then $\mathcal{T}\mathfrak{g}/I$, where I is the ideal consisting of elements $X \otimes Y + Y \otimes X - [X, Y]$ for all $X, Y \in \mathfrak{g}$.

3.2. **Proof of Schur-Weyl Duality.**

Lemma 3.1. *The image of* $\mathcal{U}(\mathfrak{gl}(V))$ *in* $\mathrm{End}(V^{\otimes n})$ *is* $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$.

Proof. The action of an element X in $\mathfrak{gl}(V)$ on $V^{\otimes n}$ is given by

$$X(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes Xv_i \otimes \cdots \otimes v_n.$$

The image of X in $\mathrm{End}(V)$ is simply the element $\Pi_n(X) \in \mathrm{End}(V^{\otimes n})$ which replicates this operation. To write it out explicitly,

$$\Pi_n(X) = \sum_{i=1}^{n} \underbrace{\mathbb{1} \otimes \cdots \otimes X \otimes \cdots \otimes \mathbb{1}}_{\substack{X \text{ in the} \\ i\text{-th position}}}.$$

This is contained in $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$ because it respects the right permutation action.

Recall that the elementary symmetric polynomial $x_1 x_2 \cdots x_n$ is representable as a polynomial in power sum symmetric polynomials $x_1^j + \cdots + x_n^j$. We can apply this identity here to express $X^{\otimes n}$ as a polynomial in $\Pi_j(X)$. Thus elements of the form $X^{\otimes n}$ are generated by elements of $\mathrm{End}(V)$ of the form $\Pi_j(X)$, which are exactly the images of $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V)$. The set spanned by elements of the form $X^{\otimes n}$ is

$$\mathrm{Sym}^n \mathrm{End}(V) \cong (\mathrm{End}(V)^{\otimes n})^{S_n} \cong \mathrm{End}(V^{\otimes n})^{S_n} = \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}),$$

so the image of $\mathcal{U}(\mathfrak{gl}(V))$ in $V^{\otimes n}$ is $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$. ■

Lemma 3.2. *The span of the images of* $\mathfrak{gl}(V)$ *and* $\mathrm{GL}(V)$ *in* $\mathrm{End}(V^{\otimes n})$ *are identical.*

Proof. Notice that $\mathrm{Span}(\mathrm{GL}(V))$ must be a subset of $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$ because the action of $\mathrm{GL}(V)$ commutes with S_n. To prove the reverse inclusion, note that the image of $g \in \mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ is $g^{\otimes n}$. We will show that any $X \in \mathrm{End}(V)$ is in the span of elements shaped like $g^{\otimes n}$. Observe that $X + tI$ is not invertible at only finitely many $t \in \mathbb{R}$, so the polynomial $(X + tI)^{\otimes n}$ is contained in the span of $g^{\otimes n}$ elements excepting finitely many t, and by interpolation this generalizes to all t. This shows the spans are equal. ■

Now we can state the theorem which grants us Schur-Weyl duality.

Theorem 3.3 (Schur-Weyl Duality)**.** *Let V be a finite dimensional vector space over $\mathbb{C}$. Then, $V^{\otimes n}$ admits a decomposition into irreducible representations of S_n and $\mathrm{GL}(V)$ as follows:*

$$V^{\otimes n} \cong \bigoplus_{|\lambda|=n} V_\lambda \otimes \mathbb{S}_\lambda V,$$

where V_λ runs through all irreducible representations of S_n and each $\mathbb{S}_\lambda V = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ is either an irreducible representation of $\mathrm{GL}(V)$ or is zero.

Proof. As S_n is a finite group, the subalgebra spanned by its image in $\mathrm{End}(V^{\otimes n})$ is semisimple. As such, by Theorem 2.3, this decomposition exists. ■

We have not defined λ yet. It is an index associated with a certain partition of an n-element set. Its meaning will be developed in the next section.

4. Irreducible Representations of the Permutation Group

4.1. **Specht Modules.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a sequence of nondecreasing nonnegative integers. Each such sequence where $\lambda_1 + \cdots + \lambda_k = n$ uniquely determines a partition of an n-element set, and therefore uniquely determines a conjugacy class of S_n. Let $p(n)$ denote the number of possible sequences λ for a given n. Recall that the number of distinct irreducible representations of a finite group G is equivalent to the number of conjugacy classes of G. Then, the number of distinct irreducible representations of S_n is $p(n)$. What is less trivial is that each distinct λ gives rise to a unique irreducible representation of S_n.

The **Young tableaux** of the partition $(3, 2, 1)$ with its canonical labeling is given below:

<table>
<tr><td>1</td><td>2</td><td>3</td></tr>
<tr><td>4</td><td>5</td><td></td></tr>
<tr><td>6</td><td></td><td></td></tr>
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.

Giving a formal definition of Young tableaux is a bit annoying, but it suffices to say that they are shapes like that with labelings like that. Each row corresponds to a λ_i in the sequence λ, and they are arranged in decreasing order to prevent duplicates in enumeration.

Consider subgroups P_λ and Q_λ of S_n which permute the labels of the Young tableaux associated with λ. P_λ only permutes the labels within each row, and Q_λ only permutes the labels in each column. Each can be associated with an element in the group algebra $\mathbb{C}[S_n]$:

$$a_\lambda = \sum_{g \in P_\lambda} g \ ; \ b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g,$$

where $\operatorname{sgn}(g)$ is the sign of the permutation g represents. We call their product $c_\lambda = a_\lambda b_\lambda$ a *Young symmetrizer.*

Theorem 4.1. *Let c_λ be a Young symmetrizer with partition size n. Then $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is an irreducible representation of S_n, and we call V_λ a Specht module.*

Proof. We first show that V_λ is simple. Recall that c_λ is idempotent up to a scalar. Let e_λ be the true idempotent associated with c_λ. Then, the subalgebra generated by $e_\lambda \mathbb{C}[S_n] e_\lambda$ is a division ring, so e_λ cannot split orthogonally, so c_λ is primitive and therefore $\mathbb{C}[S_n]c_\lambda = V_\lambda$ is simple. Since there are exactly $p(n)$ such V_λ, these are exactly the irreducible representations of S_n. ■

Theorem 4.2 (Frobenius Formula). *Let λ be a partition of n. Let $c = (c_1, \ldots, c_n)$ represent the cycle type of $g \in S_n$, such that c_i is the number of $i-$cycles in g. Set $t_i = \lambda_i + k - i$. Then,*

$$\chi_\lambda(g) = [x^t] \left(\Delta(x) \cdot \prod_j p_j(x)^{c_j} \right),$$

where $\Delta(x)$ is the Vandermonde determinant, $p_j(x)$ are the power sums, and $[x^t]f(x)$ is the coefficient of $x_1^{t_1} \cdots x_k^{t_k}$ in f.

We omit the proof for brevity (under space constraints). We will now proceed to give examples, however.

5. Examples

Example ($\mathbb{C}^2$, $n = 2$). Take $V = \mathbb{C}^2$ with standard basis $\{e_1, e_2\}$. The partitions of 2 are $\lambda = (2)$ and $\lambda = (1, 1)$, with Young diagrams

1	2

and

1
2

The corresponding Specht modules $V_{(2)}$ and $V_{(1,1)}$ are the trivial and sign representations of S_2, respectively. Schur–Weyl duality then gives the decomposition

$$V^{\otimes 2} \ = \ (\mathbb{C}^2)^{\otimes 2} \ \cong \ V_{(2)} \otimes \operatorname{Sym}^2 V \ \oplus \ V_{(1,1)} \otimes \Lambda^2 V.$$

Since $\dim \operatorname{Sym}^2 V = \binom{2+1}{2} = 3$ and $\dim \Lambda^2 V = \binom{2}{2} = 1$, this accounts for $\dim V^{\otimes 2} = 4$. Concretely, letting τ be the nontrivial transposition in S_2, the idempotent projectors

$$P_{\text{Sym}} = \tfrac{1}{2}(1 + \tau), \qquad P_\Lambda = \tfrac{1}{2}(1 - \tau)$$

cut out the two summands. One checks that

$$\mathrm{Sym}^2 V = \mathrm{Span}\Big\{\, e_1 \otimes e_1,\ \tfrac{e_1\otimes e_2 + e_2 \otimes e_1}{2},\ e_2 \otimes e_2 \Big\}, \quad \Lambda^2 V = \mathrm{Span}\Big\{ \tfrac{e_1\otimes e_2 - e_2\otimes e_1}{2} \Big\},$$

so that

$$V^{\otimes 2} = \mathrm{im}\, P_{\mathrm{Sym}} \ \oplus\ \mathrm{im}\, P_{\Lambda},$$

in agreement with the Specht modules given above.

Example (Frobenius formula for S_2). Let $g \in S_2$ have cycle-type (c_1, c_2), where c_1 is the number of 1–cycles and c_2 the number of 2–cycles. We apply the Frobenius formula

$$\chi_\lambda(g) \ = \ [x^t]\big(\,\Delta(x) \prod_{j=1}^{2} p_j(x)^{\,c_j}\big),$$

with $p_j(x) = x_1^j + x_2^j$, $\Delta(x) = x_1 - x_2$, and

$$t_i \ = \ \lambda_i + 2 - i, \quad i = 1, 2.$$

(a) $\lambda = (2)$**.** Here $\ell(\lambda) = 1$, so effectively we take one variable x and

$$t = (t_1) = (2), \qquad \Delta = 1, \qquad \prod_j p_j(x)^{c_j} = (x)^{c_1}\,(x^2)^{c_2} = x^{\,c_1 + 2c_2}.$$

Thus

$$\chi_{(2)}(g) = \ [x^2]\, x^{\,c_1+2c_2} = \begin{cases} 1, & c_1 + 2c_2 = 2, \\ 0, & \text{otherwise,} \end{cases}$$

which indeed gives $\chi_{(2)}(\mathrm{id}) = 1$ and $\chi_{(2)}((12)) = 1$, the trivial character.

(b) $\lambda = (1, 1)$**.** Now $\ell(\lambda) = 2$ so $x = (x_1, x_2)$, and

$$t = (t_1, t_2) = (\,1 + 2 - 1,\ 1 + 2 - 2\,) = (2, 1).$$

We compute

$$f(x) \ = \ \Delta(x) \prod_j p_j(x)^{c_j} = (x_1 - x_2)\,(x_1 + x_2)^{c_1}\,(x_1^2 + x_2^2)^{c_2}.$$

- For $g = \mathrm{id}$, $(c_1, c_2) = (2, 0)$, so
$$f(x) = (x_1 - x_2)(x_1 + x_2)^2 = x_1^3 + x_1^2 x_2 - x_1 x_2^2 - x_2^3,$$
and the coefficient of $x_1^2 x_2^1$ is $+1$. Hence $\chi_{(1,1)}(\mathrm{id}) = 1$.
- For the transposition (12), $(c_1, c_2) = (0, 1)$, so
$$f(x) = (x_1 - x_2)(x_1^2 + x_2^2) = x_1^3 - x_1^2 x_2 + x_1 x_2^2 - x_2^3,$$
and the coefficient of $x_1^2 x_2^1$ is -1. Thus $\chi_{(1,1)}((12)) = -1$, the sign character.

This verifies by direct coefficient-extraction that the Specht characters for $\lambda = (2)$ and $\lambda = (1, 1)$ are exactly the trivial and sign characters of S_2.

References

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[3] James Stevens. Schur–weyl duality. REU paper, University of Chicago, 2016.